Physics GRE:

Electromagnetism

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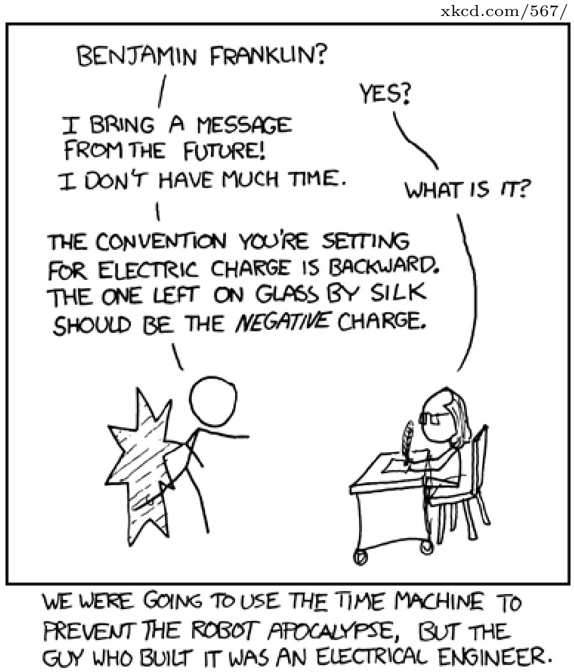

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1 Electrostatics

Coulomb's law gives the force felt between two charged particles:

$$\boldsymbol{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{\boldsymbol{r}} \qquad \boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \tag{1}$$

The electric field is the force felt per unit charge, and is defined at every point in space. For a single point charge the field is

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}} \tag{2}$$

The electric field due to a collection of particles of charges q_i and positions $\boldsymbol{r}_i$ is then

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\boldsymbol{r}-\boldsymbol{r}_i|^2}\frac{\boldsymbol{r}-\boldsymbol{r}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|} \tag{3}$$

We may also define an electric potential V by

$$\boldsymbol{E} = -\boldsymbol{\nabla} V \tag{4}$$

Note that this is *not* a potential energy, but rather has units of $^{\mathrm{J}}/_{\mathrm{C}}$. It is a useful convention to take the potential at infinity to be zero, when possible. That is,

$$V(\boldsymbol{r}) = -\int_\infty^{\boldsymbol{r}} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{s} \tag{5}$$

The use of the electric potential is that it is a scalar quantity, simplifying calculations. In electrostatics it is assumed that no charges are moving, which means that within conductors the electric field is identically zero and the electric potential is constant.

The electric field due to any charge configuration may be found using

$$\mathrm{d}\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{\mathrm{d}q}{r^2}\hat{\boldsymbol{r}} \tag{6}$$

and integrating these contributions over the charge distribution.

Gauss' law provides a useful tool for calculating the electric field of symmetric charge configurations:

$$\oint_S \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{A} = \frac{Q_{\mathrm{enc}}}{\epsilon_0} \tag{7}$$

For example, inside a sphere of radius R with uniform charge-density, ρ, we may leverage the spherical symmetry to arrive at

$$\boldsymbol{E}(\boldsymbol{r}) = \begin{cases} \frac{\rho r}{3\epsilon_0}\hat{\boldsymbol{r}} & r < R \\ \frac{\rho R^3}{3\epsilon_0 r^2}\hat{\boldsymbol{r}} & r > R \end{cases} \tag{8}$$

Expressed in terms of the total charge of the sphere, Q, this is

$$\boldsymbol{E}(\boldsymbol{r}) = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}\hat{\boldsymbol{r}} & r < R \\ \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\boldsymbol{r}} & r > R \end{cases} \tag{9}$$

We see that Gauss' law reproduces Coulomb's law for point charges.

Consider now an infinite sheet with surface charge density σ. Imagine a cylindrical Gaussian surface with faces parallel to the sheet and a distance z to either side. Applying Gauss' Law gives

$$\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A} = \frac{Q_{\text{enc}}}{\epsilon_0} \qquad \Longrightarrow \qquad E(2A) = \frac{\sigma A}{\epsilon_0} \qquad \Longrightarrow \qquad E = \frac{\sigma}{2\epsilon_0} \tag{10}$$

The electric field is constant and points away from the surface on each side. When two such sheets of opposite charge are placed parallel to one another the electric field has magnitude $E = \frac{\sigma}{\epsilon_0}$ within the gap and zero outside. This is the basis for parallel-plate capacitors.

2 Magnetostatics

The magnetic force on a moving charged particle is

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B} \tag{11}$$

The work done by this force is identically zero:

$$\mathrm{d}W = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = q(\boldsymbol{v} \times \boldsymbol{B}) \cdot (\boldsymbol{v}\,\mathrm{d}t) = 0 \tag{12}$$

This means that in the presence of only a magnetic field the kinetic energy of a particle will not change. In addition, since the force always acts perpendicular to the velocity, it causes particles to move in circular paths with the magnetic force serving as the centripetal force. For purely circular motion in a plane this gives

$$qvB = \frac{mv^2}{r} \qquad \Longrightarrow \qquad r = \frac{mv}{qB} \tag{13}$$

The radius of the circle traced out depends on the mass-charge ratio of the particle as well as its speed. In a mass spectrometer this is leveraged by selecting out a specific velocity and measuring the relative amounts in a sample based on this mass-charge ratio. Related is the cyclotron frequency, which gives the angular frequency for the circular motion that is undergone:

$$\omega = \frac{qB}{m} \tag{14}$$

For a steady current a more convenient form for the magnetic force is

$$\boldsymbol{F} = I\boldsymbol{l} \times \boldsymbol{B} \tag{15}$$

To calculate the magnetic field there is the Biot-Savart law:

$$\mathrm{d}\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{I\,\mathrm{d}\boldsymbol{l} \times \hat{\boldsymbol{r}}}{r^2} \tag{16}$$

This is mostly used for complicated geometries that do not have enough symmetry to utilize Ampère's law:

$$\oint_C \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = \mu_0 I_{\text{enc}} \tag{17}$$

For example, the cylindrical symmetry of an infinite, straight current simplifies the line integral to give

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\theta}} \tag{18}$$

3 Method of Images

When grounded conductors are involved one may use the technique of method of images to aid in the solving of electrostatics problems. Charges will arrange themselves on the conductor so as to maintain a constant potential. If enough symmetry is present then this may be equivalent to considering "image" charges to give the same boundary conditions on the potential. The simplest example is a point charge above a conducting sheet: the configuration is equivalent to having an oppositely charged particle "mirrored" on the opposite side of the sheet.

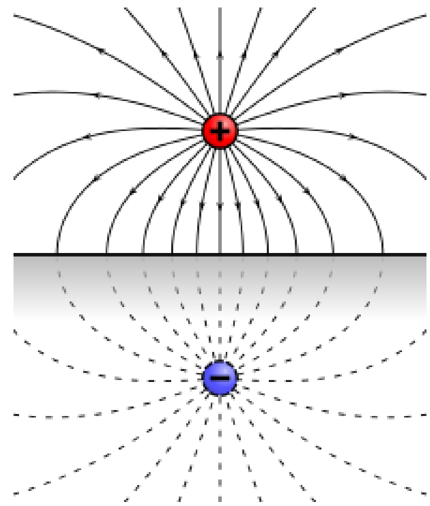

Figure 1: Method of images for point charge and conducting sheet. This is equivalent to having the two point charges shown.

4 Lorentz Force

In regions with both electric and magnetic fields the total force is known as the Lorentz force:

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{19}$$

A moving particle feels zero net force if its direction of motion, the electric field and magnetic field are all pairwise orthogonal, and the magnitudes of the three are related by

$$|E| = |v| \cdot |B| \tag{20}$$

This is a convenient mnemonic to remember the relationship between the units for electric and magnetic fields:

$$[\boldsymbol{E}] = [c][\boldsymbol{B}] \tag{21}$$

5 Maxwell's Equations

Let Ω be a region with closed surface $\partial\Omega$ and let A be a surface with closed boundary ∂A. Then Maxwell's equations may be written:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \qquad \oint_{\partial\Omega} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A} = \frac{1}{\epsilon_0} \int_{\Omega} \rho \, \mathrm{d}V \tag{22}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \qquad \oint_{\partial\Omega} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{A} = 0 \tag{23}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \oint_{\partial A} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{A} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{A} \tag{24}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \qquad \oint_{\partial A} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 \int_{A} \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{A} + \mu_0 \epsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_{A} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A} \tag{25}$$

5.1 Electromagnetic Four-Potential Expressing $\boldsymbol{E}$ and $\boldsymbol{B}$ in terms of the electric potential and vector potential gives

$$\boldsymbol{E} = -\boldsymbol{\nabla} V - \frac{\partial \boldsymbol{A}}{\partial t} \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \tag{26}$$

This changes the form of Maxwell's equations considerably. Two are automatically satisfied by our choice of $\boldsymbol{E}$ and $\boldsymbol{B}$, and the remaining two become

$$\nabla^2 V + \frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) = -\frac{\rho}{\epsilon_0} \qquad \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} + \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right) = \mu_0 \boldsymbol{J} \tag{27}$$

We may make a gauge transformation

$$V \to V - \frac{\partial \phi}{\partial t} \qquad \boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\phi \tag{28}$$

without changing the observables, namely the fields. In practice this means we may choose a constraint for $\boldsymbol{\nabla} \cdot \boldsymbol{A}$. Two common gauges are the Coulomb gauge and the Lorenz gauge.

- Coulomb Gauge: $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \qquad \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\boldsymbol{\nabla} V = \mu_0 \boldsymbol{J} \tag{29}$$

  This has the advantage of cleaning up the first equation.

- Lorenz Gauge: $\boldsymbol{\nabla} \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$

$$\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \nabla^2 V = \frac{\rho}{\epsilon_0} \qquad \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J} \tag{30}$$

  This clearly puts the potentials on the same footing and shows that with no sources they satisfy the wave equation.

5.2 Auxiliary Fields When not in a vacuum it is sometimes convenient to introduce two auxiliary fields:

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} \qquad \boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M} \tag{31}$$

where $\boldsymbol{P}$ is the polarization and M is the magnetization. In a linear material these are simply proportional to $\boldsymbol{E}$ and $\boldsymbol{B}$, and so in the simplest of cases we have

$$\boldsymbol{D} = \epsilon \boldsymbol{E} \qquad \boldsymbol{H} = \frac{1}{\mu}\boldsymbol{B} \tag{32}$$

where ϵ is the permittivity and μ is the permeability of the material, both of which may depend on position and time. We also have

$$\frac{\epsilon}{\epsilon_0} = \epsilon_r = 1 + \chi_e \qquad \frac{\mu}{\mu_0} = \mu_r = 1 + \chi_m \tag{33}$$

where χ_e and χ_m are the electric and magnetic susceptibilities.

In terms of these auxiliary fields, Maxwell's equations become

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_{\mathrm{f}} \tag{34}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{35}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{36}$$

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_{\mathrm{f}} + \frac{\partial \boldsymbol{D}}{\partial t} \tag{37}$$

where ρ_{f} is the free charge density given by $-\boldsymbol{\nabla} \cdot \boldsymbol{P}$ and $\boldsymbol{J}_{\mathrm{f}}$ is the free current density given by $\boldsymbol{\nabla} \times \boldsymbol{M} + \frac{\partial \boldsymbol{P}}{\partial t}$.

5.3 Boundary Conditions At the interface of media the electric and magnetic fields satisfy several boundary conditions. These are

$$\epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp = D_1^\perp - D_2^\perp = \sigma_f \tag{38}$$

$$B_1^\perp - B_2^\perp = 0 \tag{39}$$

$$\boldsymbol{E}_1^\parallel - \boldsymbol{E}_2^\parallel = \boldsymbol{0} \tag{40}$$

$$\frac{1}{\mu_1}\boldsymbol{B}_1^\parallel - \frac{1}{\mu_2}\boldsymbol{B}_2^\parallel = \boldsymbol{H}_1^\parallel - \boldsymbol{H}_2^\parallel = \boldsymbol{K}_f \times \hat{\boldsymbol{n}} \tag{41}$$

σ_f is the free charge density on the surface and $\boldsymbol{K}_f$ is the current density.

6 Electromagnetic Induction

Maxwell's equations describe electromagnetism even when the fields are time-dependent. Colloquially, changing electric fields create magnetic fields and changing magnetic fields create electric fields. Define the electric and magnetic flux through a surface by

$$\Phi_\mathrm{E} = \int_A \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A} \qquad \Phi_\mathrm{B} = \int_A \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{A} \tag{42}$$

Maxwell's equations in the absence of sources now include

$$\oint_{\partial A} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}\Phi_\mathrm{B}}{\mathrm{d}t} \qquad \oint_{\partial A} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0\epsilon_0 \frac{\mathrm{d}\Phi_\mathrm{E}}{\mathrm{d}t} \tag{43}$$

The second of these is less frequently encountered. The first shows that around a closed loop there is an emf proportional to the time-rate-of-change of the magnetic flux through the encircled surface. This emf may cause the flow of charges, known as an induced current. This induced current in turn creates a magnetic field which opposes that which created it: this is Lenz's law, manifest in the minus sign found in the equation above. In summary, the emf generated in any loop, regardless of whether there are charges or not, is given by

$$\mathcal{E} = -\frac{\mathrm{d}\Phi_\mathrm{B}}{\mathrm{d}t} \tag{44}$$

The inductance is the constant of proportionality between the magnetic flux and the current which induces it. Mutual inductance, M, is between different circuit elements and self inductance, L, characterizes self-interactions. For a solenoid with cross-sectional area A and turn density n the magnetic field and self-inductance are

$$B = \mu_0 n I \qquad L = \mu_0 n^2 A l \tag{45}$$

Notice that L depends only on the geometry of the solenoid; this is a general result.

Consider a square loop of wire near a infinite straight wire with steady current I. The loop has side-length s and a resistance R, and begins a distance d_0 from the wire. The task is to find the current as a function of time in the loop as it is pulled away from the wire at a constant speed v. Begin by finding the magnetic flux through the loop as a function of time:

$$\Phi_\mathrm{B} = \int \mathrm{d}\Phi_\mathrm{B} = \int_A \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{A} = \int_A \frac{\mu_0 I}{2\pi r} s\,\mathrm{d}r = \frac{\mu_0 I s}{2\pi} \log r \Big|_{r=d(t)}^{d(t)+s} = \frac{\mu_0 I s}{2\pi} \log\left[\frac{d(t)+s}{d(t)}\right] \tag{46}$$

Since the distance to the wire as a function of time is $d(t) = d_0 + vt$, we have

$$\Phi_\mathrm{B} = \frac{\mu_0 I s}{2\pi} \log\left(\frac{d_0 + vt + s}{d_0 + vt}\right) \tag{47}$$

This gives the following induced EMF in the loop:

$$\mathcal{E} = -\frac{\mathrm{d}\Phi_\mathrm{B}}{\mathrm{d}t} = -\frac{\mu_0 I s}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t} \log\left(\frac{d_0 + vt + s}{d_0 + vt}\right) = \frac{\mu_0 I s^2 v}{2\pi}\frac{1}{(d_0 + vt)(d_0 + vt + s)} \tag{48}$$

This gives for the induced current

$$I_\mathrm{in} = \frac{\mathcal{E}}{R} = \frac{\mu_0 I s^2 v}{2\pi R}\frac{1}{(d_0 + vt)(d_0 + vt + s)} \tag{49}$$

7 Electromagnetic Waves

Maxwell's equations in a vacuum, where $\rho = 0$ and $\boldsymbol{J} = \boldsymbol{0}$, reduce to

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0 \qquad \boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{50}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \qquad \boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \tag{51}$$

Some vector calculus reshuffling decouples these equations to give

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \nabla^2 \boldsymbol{E} = 0 \qquad \frac{1}{c^2}\frac{\partial^2 \boldsymbol{B}}{\partial t^2} - \nabla^2 \boldsymbol{B} = 0 \qquad c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{52}$$

Both the electric and magnetic fields satisfy the wave equation with the phase velocity c. The electric field, magnetic field and direction of propagation are all pairwise orthogonal.

When not in vacuum the phase velocity of light changes to

$$v = \frac{1}{\sqrt{\mu \epsilon}} = \frac{1}{\sqrt{\mu_r \mu_0 \epsilon_r \epsilon_0}} = \frac{c}{n} \qquad n = \sqrt{\mu_r \epsilon_r} \tag{53}$$

where n is the index of refraction, μ_r is the relative permeability and ϵ_r is the relative permittivity.

8 Circuits

8.1 Circuit Elements Due to the large number of conducting electrons in a metal only a very slight drift velocity results in huge currents. If n denotes the number density of charge carriers, q their individual charge, A the cross-sectional area of the wire and I the measured current, then the drift velocity of the charge carriers is

$$u = \frac{I}{nAq} \qquad \frac{1\,\mathrm{A}}{(10^{29}\,\mathrm{m}^{-3})(10^{-6}\,\mathrm{m}^2)(10^{-19}\,\mathrm{C})} = 10^{-4}\,\mathrm{m/s} \tag{54}$$

The above shows approximate values for the drift velocity in a thin copper wire. Notice how slow this is compared to the average speed of an electron.

Current is the rate at which charge passes a point:

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t} \tag{55}$$

Resistance is measured in Ohms, and objects for which the potential and current are directly proportional are said to be Ohmic. The resistance is the constant of proportionality:

$$V = IR \tag{56}$$

This is an idealization for many objects that holds well at low currents. Things such as diodes and batteries in no way follow this relationship. The power dissipated by Ohmic devices is

$$P = IV = I^2 R = \frac{V^2}{R} \tag{57}$$

Capacitors have a potential proportional to the charge separated. In the case of a parallel-plate capacitor the electric field is constant and one finds that

$$V = \frac{Qd}{\epsilon_0 A} \qquad \Longrightarrow \qquad Q = \left(\frac{\epsilon_0 A}{d}\right) V \tag{58}$$

In general we have

$$Q = CV \tag{59}$$

We see that increasing the area of the parallel plates or decreasing their separation distance increases the capacitance. The energy stored in the electric field by a capacitor is

$$U = \frac{1}{2} C V^2 = \frac{1}{2} Q V = \frac{Q^2}{2C} \tag{60}$$

The voltage across an inductor is given by

$$V = L \, \frac{\mathrm{d}I}{\mathrm{d}t} \tag{61}$$

The energy stored in the magnetic field by an inductor is

$$U = \frac{1}{2} L I^2 \tag{62}$$

Resistors and capacitors may be combined in parallel or parallel; such combination may be thought of as single resistors or capacitors with an "effective" resistance or capacitance.

$$R_{\text{ser}} = \sum_i R_i \qquad\qquad \frac{1}{C_{\text{ser}}} = \sum_i \frac{1}{C_i} \tag{63}$$

$$\frac{1}{R_{\text{par}}} = \sum_i \frac{1}{R_i} \qquad\qquad C_{\text{par}} = \sum_i C_i \tag{64}$$

Since changes in the electric potential are independent of path, the sum of potential changes around closed loops must vanish. This and conservation of current constitute Kirchhoff's circuit laws:

- Around closed loops we have $\sum_i V_i = 0$.
- At each junction we have $\sum_i I_i = 0$, where the algebraic sign of the current distinguishes in- and out-going currents.

Applying these rules to a circuit gives a linear system of equations in the unknowns, lending itself nicely to linear algebra techniques.

8.2 RC Circuits A resistor and capacitor may be connected in series to a battery to charge the capacitor. Applying Kirchhoff's laws to such a circuit results in

$$V - RI - \frac{Q}{C} = 0 \qquad \Longrightarrow \qquad \frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{RC} - \frac{V}{R} = 0 \tag{65}$$

Solving this for an initial charge $Q(0) = 0$ gives

$$Q(t) = CV\left[1 - \exp\left(-\frac{t}{RC}\right)\right] \qquad I(t) = \frac{V}{R}\exp\left(-\frac{t}{RC}\right) \tag{66}$$

The quantity RC is known as the time-constant for the circuit.

A charged capacitor may be discharged through a resistor. The defining equation and its solution for an initial charge Q_0 are

$$RI + \frac{Q}{C} = 0 \qquad Q(t) = Q_0\exp\left(-\frac{t}{RC}\right) \qquad I(t) = -\frac{Q_0}{RC}\exp\left(-\frac{t}{RC}\right) \tag{67}$$

The equations here are analogous to a body in free-fall with a linear drag force under the association $Q \leftrightarrow v$.

8.3 RL Circuits Connecting a resistor and inductor in series across a battery gives the following voltage equation

$$V - RI - L\frac{\mathrm{d}I}{\mathrm{d}t} = 0 \qquad \Longrightarrow \qquad \frac{\mathrm{d}I}{\mathrm{d}t} + \frac{RI}{L} - \frac{V}{L} = 0 \tag{68}$$

Solving this for an initial current $I(0) = 0$ gives

$$I(t) = \frac{V}{R}\left[1 - \exp\left(-\frac{t}{R/L}\right)\right] \tag{69}$$

The quantity $\frac{R}{L}$ is the time-constant for the circuit.

8.4 LC Circuits A circuit consisting of a capacitor and inductor is described by

$$L\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{Q}{C} = 0 \qquad \Longrightarrow \qquad \frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + \frac{Q}{LC} = 0 \tag{70}$$

This is the equation for simple harmonic motion with frequency $\omega = \frac{1}{\sqrt{LC}}$, and so the solution with initial conditions $Q(0) = Q_0$ and $I(0) = I_0$ is

$$Q(t) = Q_0\cos\left(\frac{t}{\sqrt{LC}}\right) + I_0\sqrt{LC}\sin\left(\frac{t}{\sqrt{LC}}\right) \tag{71}$$

The energy of the system is not dissipated by any resistors, and so remains constant.

8.5 LRC Circuits When a resistor, inductor and charge capacitor are connected in series the system is described by

$$RI + L\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{Q}{C} = 0 \qquad \Longrightarrow \qquad \frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{LC} = 0 \tag{72}$$

This is the equation for damped harmonic motion with natural frequency $\omega = \frac{1}{\sqrt{LC}}$. The nature of the solutions is exactly the same as those for a damped harmonic oscillator with the identification $Q \leftrightarrow x$. See the Classical notes for a discussion of the solutions.

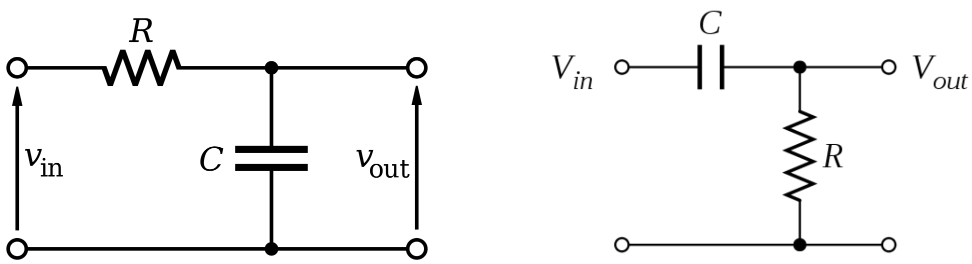

Figure 2: Examples of low-pass (left) and high-pass (right) filters.

8.6 High- & Low-Pass Filters A high- or low-pass filter uses properties of resistors, capacitors and inductors to selectively allow through high or low frequency signals in AC circuits.

The impedance for a resistor, capacitor and inductor are

$$Z_{\mathrm{R}} = R \qquad Z_{\mathrm{C}} = \frac{1}{i\omega C} \qquad Z_{\mathrm{L}} = i\omega L \tag{73}$$

Any impedance may be written $Z = R + iX$, where R is the resistance and X is the reactance. Reactance is similar to resistance in that it represents an opposition to changes in voltage or current. To determine if a given configuration is a high- or low-pass filter, look at the impedance for large and small values of ω. For example, at high frequencies we have $Z_{\mathrm{C}} \to 0$ so that any capacitor acts as a wire, and at low frequencies we have $Z_{\mathrm{C}} \to \infty$ so that it now acts as a cut wire. Using such limiting behaviours one can determine which frequencies output nonzero potential differences.

A Summary

Electrostatics

$$\boldsymbol{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{\boldsymbol{r}} \qquad \text{(Coulomb's Law)}$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{\boldsymbol{F}}{q} = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\boldsymbol{r}-\boldsymbol{r}_i|^2}\frac{\boldsymbol{r}-\boldsymbol{r}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|} \qquad \text{(Electric Field)}$$

$$\boldsymbol{E} = -\boldsymbol{\nabla} V \qquad \text{(Electric Potential)}$$

$$V(\boldsymbol{r}) = -\int_{\infty}^{r} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{s}$$

Magnetostatics

$$\boldsymbol{F} = q\boldsymbol{v}\times\boldsymbol{B} = I\boldsymbol{l}\times\boldsymbol{B} \qquad \text{(Magnetic Force)}$$

$$\omega = \frac{qB}{m} \qquad \text{(Cyclotron Frequency)}$$

$$\mathrm{d}\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{I\,\mathrm{d}\boldsymbol{l}\times\hat{\boldsymbol{r}}}{r^2} \qquad \text{(Biot-Savart Law)}$$

Maxwell's Equations

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{\rho}{\epsilon_0} \qquad \text{(Gauss' Law)}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \qquad \text{(Gauss' Law for Magnetism)}$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} \qquad \text{(Faraday's Law)}$$

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0\boldsymbol{J} + \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t} \qquad \text{(Ampère's Law)}$$

$$\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P} = \epsilon\boldsymbol{E} = \epsilon_0(1+\chi_e)\boldsymbol{E} \qquad \text{(Auxillary Fields)}$$

$$\boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M} = \frac{1}{\mu}\boldsymbol{B} = \frac{1}{\mu_0(1+\chi_m)}\boldsymbol{B}$$

Electromagnetic Induction

$$\Phi_{\mathrm{E}} = \int_A \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{A} \qquad \text{(Electric Flux)}$$

$$\Phi_{\mathrm{B}} = \int_A \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{A} \qquad \text{(Magnetic Flux)}$$

$$\mathcal{E} = -\frac{\mathrm{d}\Phi_{\mathrm{B}}}{\mathrm{d}t} \qquad \text{(Induced EMF)}$$

$$L = \mu_0 n^2 A l \qquad \text{(Solenoid Self-Inductance)}$$

Electromagnetic Waves

$$\frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2} = \nabla^2\boldsymbol{E} \qquad \text{(Vacuum Wave Equations)}$$

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{B}}{\partial t^2} = \nabla^2 \boldsymbol{B}$$

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \qquad \text{(Speed of Light)}$$

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_r \mu_0 \epsilon_r \epsilon_0}} = \frac{c}{\sqrt{\mu_r \epsilon_r}} = \frac{c}{n} \qquad \text{(Speed of Light in Medium)}$$

Circuits

$$u = \frac{I}{nAq} \qquad \text{(Drift Velocity)}$$

$$V = IR \qquad \text{(Ohm's Law)}$$

$$P = IV = I^2 R = \frac{V^2}{R} \qquad \text{(Disspated Power)}$$

$$Q = CV \qquad \text{(Capacitance)}$$

$$U = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{Q^2}{2C} \qquad \text{(Energy of Capacitor)}$$

$$V = L\frac{\mathrm{d}I}{\mathrm{d}t} \qquad \text{(Inductor Potential)}$$

$$U = \frac{1}{2}LI^2 \qquad \text{(Energy of Inductor)}$$

$$R_{\text{ser}} = \sum_i R_i \qquad \text{(Effective Resistance)}$$

$$\frac{1}{R_{\text{par}}} = \sum_i \frac{1}{R_i}$$

$$\frac{1}{C_{\text{ser}}} = \sum_i \frac{1}{C_i} \qquad \text{(Effective Capacitance)}$$

$$C_{\text{par}} = \sum_i C_i$$

$$\oint_C V \, \mathrm{d}l = 0 \qquad \text{(Kirchhoff Loop Rule)}$$

$$\sum_i I_i = 0 \qquad \text{(Kirchhoff Current Rule)}$$

$$\tau = RC \qquad \text{(RC Time Constant)}$$

$$\tau = \frac{R}{L} \qquad \text{(RL Time Constant)}$$

$$\omega = \frac{1}{\sqrt{LC}} \qquad \text{(LC Frequency)}$$

$$Z_{\text{C}} = \frac{1}{i\omega C} \qquad \text{(Capacitor Impedance)}$$

$$Z_{\text{L}} = i\omega L \qquad \text{(Inductor Impedance)}$$