Physics GRE:
Optics & Wave Phenomena

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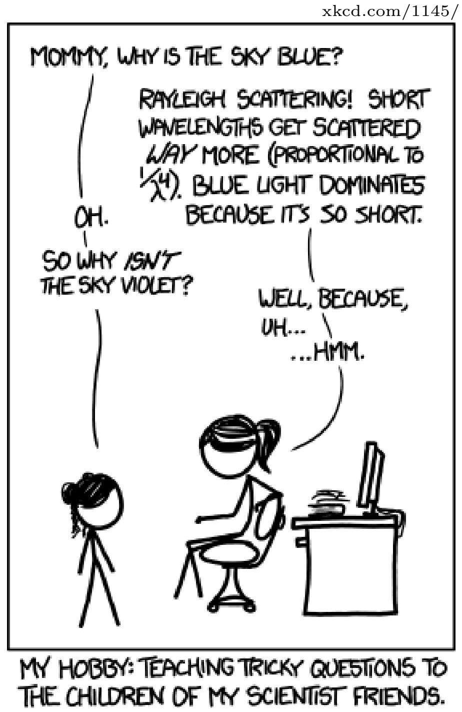

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1 Wave Properties

The simplest wave is of the form

$$\cos kx = \cos\left(\frac{2\pi x}{\lambda}\right) \qquad k = \frac{2\pi}{\lambda} \tag{1}$$

where k is the wavenumber and λ is the wavelength. The wavelength is the distance between subsequent peaks. One may also consider a waveform which changes over time, such as

$$\cos(kx \mp \omega t) = \cos\left[2\pi\left(\frac{x}{\lambda} \mp \frac{t}{T}\right)\right] \qquad \omega = \frac{2\pi}{T} \tag{2}$$

where T is the period of oscillation. A minus sign corresponds to a wave travelling towards increasing x, and a plus sign corresponds to a wave travelling towards decreasing x. The space and time derivatives of such a wave are related by

$$\frac{1}{v_{\mathrm{p}}^2}\frac{\partial^2\phi}{\partial t^2} = \nabla^2\phi \qquad v_{\mathrm{p}} = \frac{\omega}{k} = \frac{\lambda}{T} \tag{3}$$

This is the wave equation. This is a linear equation, and so any linear combination of solutions is also a solution: this is the principle of superposition. The constant v_{p} is the phase velocity, giving the velocity for the peaks in the waveform. There is also the group velocity, given by

$$v_{\mathrm{g}} = \frac{\mathrm{d}\omega(k)}{\mathrm{d}k} \tag{4}$$

which arises when considering wave packets of the form

$$\phi(x;t) = \int_{-\infty}^{\infty} \mathrm{d}k\ \alpha(k) e^{i[kx-\omega(k)t]} \tag{5}$$

where α is a function of k, sharply peaked near some k_0. For different functions α the wave obtains different "envelopes" which move with velocity v_{g}. Note that the phase and group velocities may be of opposite sign, indicating that the individual peaks and envelopes are moving in opposite directions. These waves generalize to multiple dimensions with the following changes:

$$k \to \boldsymbol{k} \qquad \boldsymbol{v}_{\mathrm{p}} = \frac{\omega}{|\boldsymbol{k}|}\hat{\boldsymbol{k}} \qquad \boldsymbol{v}_{\mathrm{g}} = \boldsymbol{\nabla}_{\boldsymbol{k}}\,\omega \tag{6}$$

2 Beat Patterns

If two waves of similar frequency interfere then they produce "beats". Using trigonometric identities, we have

$$A\cos(\omega_1 t) + A\cos(\omega_2 t) = 2A\cos\left[\left(\frac{\omega_1-\omega_2}{2}\right)t\right]\cos\left[\left(\frac{\omega_1+\omega_2}{2}\right)t\right] \tag{7}$$

The superposition of two waves gives a wave with frequency the average of the two within an envelope giving the beats. The beat frequency is given by

$$2\pi f_{\mathrm{B}} = \omega_{\mathrm{B}} = |\omega_1 - \omega_2| \tag{8}$$

This differs by a factor of two from one might expect. This choice in definition arises because the envelope goes to zero twice per period.

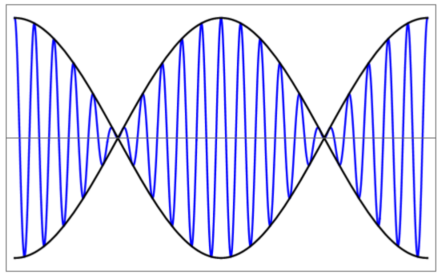

Figure 1: Beat pattern showing the sinusoidal envelope.

3 Interference

When two waves combine they are said to interfere. When they cancel each other out this is destructive interference and when they add this is constructive interference. A good example of these phenomena is through the double-slit experiment: a plane wave is made to pass through two thin slits and strike a distance screen. An interference pattern arises on the screen due to differing path lengths from each slit; if the difference is an integer multiple of the wavelength then there is constructive interference and a bright spot is observed, and if the difference is a half-integer multiple of the wavelength then there is deconstructive interference and the screen there is dark. For a wavelength λ, slit separation d, distance to screen D and distance from center of screen y, geometrical considerations lead to the following condition for a maximum in the observed intensity:

$$n\lambda = d\sin\theta \approx d \cdot \frac{y}{D} \qquad n \in \mathbb{Z} \tag{9}$$

Minima occur where

$$\left(n + \tfrac{1}{2}\right)\lambda = d\sin\theta \approx d \cdot \frac{y}{D} \qquad n \in \mathbb{Z} \tag{10}$$

4 Diffraction

When a wave meets an obstacle or slit it tends to “bend” around the edges of the object. This is readily seen when plane waves pass through a single slit or through a circular aperture. These lead to diffraction patterns of bright and dark spots that arise from the wave interfering with itself. Similar treatment as in the case of two slits gives the following condition for peak intensity for a single slit diffraction

$$\left(n + \tfrac{1}{2}\right)\lambda = a\sin\theta \approx a \cdot \frac{y}{D} \qquad n \in \mathbb{Z} \tag{11}$$

where a is the width of the slit. The locations of the minima are where

$$n\lambda = a\sin\theta \approx a \cdot \frac{y}{D} \qquad n \in \mathbb{Z} \tag{12}$$

Note that the correspondence between maxima, minima, integers and half-integers is opposite as for the case of double-slit interference.

Of course, in the double-slit experiment one should also account for the diffraction arising from each of the slits individually. The result is the double-slit interference pattern within the envelope of the single-slit diffraction. The intensity on the screen is given by

$$I = I_0\cos^2\left(\frac{\pi d\sin\theta}{\lambda}\right)\operatorname{sinc}^2\left(\frac{\pi a\sin\theta}{\lambda}\right) \approx I_0\cos^2\left(\frac{\pi d\, y}{\lambda D}\right)\operatorname{sinc}^2\left(\frac{\pi a\, y}{\lambda D}\right) \qquad \operatorname{sinc} x = \frac{\sin x}{x} \tag{13}$$

The cosine provides the interference pattern within the sinc envelope.

5 Standing Waves & Thin Films

In one dimension, waves travelling in opposite directions may interfere to produce standing waves:

$$A \sin(kx - \omega t) + A \sin(kx + \omega t) = 2A \sin kx \cos \omega t \tag{14}$$

Objects such as violin strings and organ pipes with "open" and/or "closed" ends allow standing waves. At a closed end there must be a node and at an open end there must be an anti-node. In a pipe of length L which is half-open, one sees the pipe can allow standing waves such that the following holds:

$$L = \frac{2n+1}{4}\lambda_n \qquad \Longrightarrow \qquad \lambda_n = \frac{4L}{2n+1} \qquad n \in \{0, 1, 2, \ldots\} \tag{15}$$

With speed of sound c, these correspond to the frequencies

$$f_n = \frac{c}{\lambda_n} = \frac{(2n+1)c}{4L} = (2n+1)f_0 \qquad n \in \{0, 1, 2, \ldots\} \tag{16}$$

where f_0 is known as the fundamental. All other frequencies are odd multiples of the fundamental.

In the case of two ends of the same type, the object allows the following wavelengths and corresponding frequencies:

$$L = \frac{n}{2}\lambda_n \qquad \Longrightarrow \qquad \lambda_n = \frac{2L}{n} \qquad f_n = \frac{c}{\lambda_n} = \frac{nc}{2L} = nf_1 \qquad n \in \{1, 2, 3, \ldots\} \tag{17}$$

Here f_1 denotes the fundamental, and all other allowed frequencies are integer multiples of f_1.

The discussion of thin film interference bears some resemblance to the above analysis. When light is incident upon a region with index of refraction n_2 sandwiched between regions with indices n_1 and n_3, as is seen in Figure 2, the waves which reflect by different paths may interfere either constructively or deconstructively. For light which strikes the interface head-on, the difference in path-length for the two paths shown is given by $\Delta x = 2d$, where d is the thickness of the thin film. When $n_3 > n_2$ phase of the wave changes by π and is unaffected when $n_3 < n_2$. This change of phase may also occur for the reflection off of the top of the film. There are no phase changes for transmitted waves. Going forward, assume that $n_1 = 1$ for simplicity. This means that there is a change of phase of π for the initially reflected path. In the case where $n_3 > n_2$, there is also a phase change of π for the second path, and so maxima will occur when the difference in path length is an integer multiple of the wavelength in the thin film:

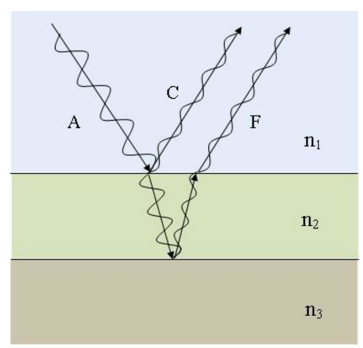

Figure 2: Thin film geometry.

$$2d = m \cdot \frac{\lambda}{n_2} \qquad m \in \{1, 2, 3, \ldots\} \tag{18}$$

Of course, the minima occur where

$$2d = \left(m + \frac{1}{2}\right)\frac{\lambda}{n_2} \qquad m \in \{0, 1, 2, \ldots\} \tag{19}$$

When there is no phase change at the second interface, i.e. when $n_3 > n_2$, these conditions are flipped, so that the maxima occur where

$$2d = \left(m + \frac{1}{2}\right)\frac{\lambda}{n_2} \qquad m \in \{0, 1, 2, \ldots\} \tag{20}$$

and the minima occur where

$$2d = m \cdot \frac{\lambda}{n_2} \qquad m \in \{1, 2, 3 \ldots\} \tag{21}$$

6 Geometrical Optics

In this section the Cartesian sign convention will be used: the positive direction for all quantities is the direction of light propagation.

When light passes from a region of one index of refraction to another it will reflect off the surface such that the angle made from the normal is the same for incident and reflected rays. When refracting, the two angles from the normal are related through Snell's law:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{22}$$

At the critical angle all incident light will we reflected. This occurs when $\sin\theta_2 = 1$; that is, where

$$\theta_{\text{crit}} = \arcsin\left(\frac{n_2}{n_1}\right) \tag{23}$$

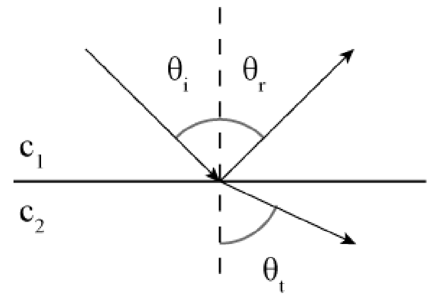

Figure 3: Geometry for reflection and refraction.

In going from water ($n \approx 1.33$) to air ($n \approx 1$) the critical angle is $\theta_{\text{crit}} \approx 48.6°$. Another important angle in this context is Brewster's angle, at which the light that is reflected off the surface is perfectly polarized. It is given by

$$\theta_{\text{Brew}} = \arctan\left(\frac{n_2}{n_1}\right) \tag{24}$$

The lensmaker's equation gives the focal length of a thin lens based on its index of refraction and the curvatures of its two surfaces:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{25}$$

The thin lens equation relates the positions of an object, the focal length and the image:

$$\frac{1}{o} + \frac{1}{f} = \frac{1}{i} \tag{26}$$

With a series of lenses the image of each serves as the object for the next.

The magnification of a lens is given by

$$M = \frac{i}{o} \tag{27}$$

A mirror has a focal length related to its radius of curvature by

$$f = -\frac{r}{2} \tag{28}$$

A derivation of this relies on a small angle approximation.

7 Light Polarization

As mentioned in the electromagnetism notes, light is the propagation of electric and magnetic fields through the vacuum, with the two fields and the direction of propagation being pair-wise orthogonal. The polarization of light is in the direction of the electric field, and may be though of as living in the plane orthogonal to the motion. Linearly-polarized light corresponds to the electric field oscillating along a straight line, while circularly-polarized light corresponds to the electric field vector sweeping out a circle: the two directions of rotation are called left- and right-handed polarization states.

Linear polarizers may be used to select out only a portion of a beam of photons based on their polarization states. Malus' law gives the relationship between the intensities of the beam before and after passing through a polarizer:

$$I = I_0 \cos^2 \theta \tag{29}$$

Where θ is the angle between the polarization of the incident beam and the polarizer itself. If the incident beam is unpolarized, then one averages over all polarization states to find

$$I = \frac{1}{2\pi} \int_0^{2\pi} I_0 \cos^2 \theta \, \mathrm{d}\theta = \frac{I_0}{2} \tag{30}$$

That is, the intensity of an unpolarized beam is reduced by a factor of two in passing through a polarizer. When several polarizers are placed in the path of a beam, their effects are multiplied in the obvious way, with each angle being measured between the polarization of the light as it exits one polarizer and the next polarizer in the series. For example, if unpolarized light is incident on a pair of polarizers oriented at an angle of $\frac{\pi}{4}$ from each other, then the final intensity of the beam will be $I = \frac{I_0}{4}$.

8 Doppler Effect

When a wave source is moving towards to an observer then subsequent peaks in the waveform will arrive more quickly: the observed frequency has increased. In addition the wavelength has decreased, as the propagation speed is a constant about which all observers agree. If c denotes the speed of waves in a particular medium, v_s the velocity of the source and v_o the velocity of the observer, then we have

$$f = \left(\frac{c + v_o}{c + v_s} \right) f_0 \tag{31}$$

where both velocities are relative to the medium, v_s is positive when the source is moving away from the observer, and v_o is positive when the observer is moving towards the source. Notice that weird things happen when the source is moving towards the observer with $|v_s| = c$: this occurs when "breaking the sound barrier". When both velocities are small we may Taylor expand to get

$$f = \left(1 + \frac{v_o - v_s}{c} \right) f_0 = \left(1 + \frac{\Delta v}{c} \right) f_0 \qquad \Longrightarrow \qquad \Delta f = \frac{\Delta v}{c} f_0 \tag{32}$$

Keep in mind that this section is a non-relativistic treatment: c is not, as is usual, the speed of light!

A Summary

Wave Properties

$$\frac{1}{v_{\mathrm{p}}^2}\frac{\partial^2\phi}{\partial t^2} = \nabla^2\phi \qquad \text{(Wave Equation)}$$

$$v_{\mathrm{p}} = \frac{\omega}{k} = \frac{\lambda}{T} \qquad \text{(Phase Velocity)}$$

$$v_{\mathrm{g}} = \frac{\mathrm{d}\omega(k)}{\mathrm{d}k} \qquad \text{(Group Velocity)}$$

$$2\pi f_{\mathrm{B}} = \omega_{\mathrm{B}} = |\omega_1 - \omega_2| \qquad \text{(Beat Frequency)}$$

Interference & Diffraction

$$n\lambda = d\sin\theta \approx d\cdot\frac{y}{D} \qquad \text{(Double-Slit Maxima \& Single-Slit Minima)}$$

$$\left(n+\tfrac{1}{2}\right)\lambda = d\sin\theta \approx d\cdot\frac{y}{D} \qquad \text{(Double-Slit Minima \& Single-Slit Maxima)}$$

$$I = I_0\cos^2\left(\frac{\pi d\,y}{\lambda D}\right)\mathrm{sinc}^2\left(\frac{\pi a\,y}{\lambda D}\right) \qquad \text{(Double-Slit Intensity)}$$

$$f_n = \frac{c}{\lambda_n} = \frac{(2n+1)c}{4L} = (2n+1)f_0 \qquad \text{(Half-Open Frequencies)}$$

$$f_n = \frac{c}{\lambda_n} = \frac{nc}{2L} = nf_1 \qquad \text{(Both Open Freqencies)}$$

$$2d = m\cdot\frac{\lambda}{n_2} \qquad \text{(Phase Shift Maxima \& No Phase Shift Minima)}$$

$$2d = \left(m+\frac{1}{2}\right)\frac{\lambda}{n_2} \qquad \text{(Phase Shift Minima \& No Phase Shift Maxima)}$$

Geometrical Optics

$$n_1\sin\theta_1 = n_2\sin\theta_2 \qquad \text{(Snell's Law)}$$

$$\theta_{\mathrm{crit}} = \arcsin\left(\frac{n_2}{n_1}\right) \qquad \text{(Critical Angle)}$$

$$\theta_{\mathrm{Brew}} = \arctan\left(\frac{n_2}{n_1}\right) \qquad \text{(Brewster's Angle)}$$

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \qquad \text{(Lensmaker's Equation)}$$

$$\frac{1}{o}+\frac{1}{f} = \frac{1}{i} \qquad \text{(Thin Lens Equation)}$$

$$M = \frac{i}{o} \qquad \text{(Magnification)}$$

$$I = I_0\cos^2\theta \qquad \text{(Malus' Law)}$$

Doppler Effect

$$f = \left(\frac{c+v_{\mathrm{o}}}{c+v_{\mathrm{s}}}\right)f_0 \approx \left(1+\frac{\Delta v}{c}\right)f_0 \qquad \text{(Doppler Shift)}$$