Physics GRE:

Specialized Topics

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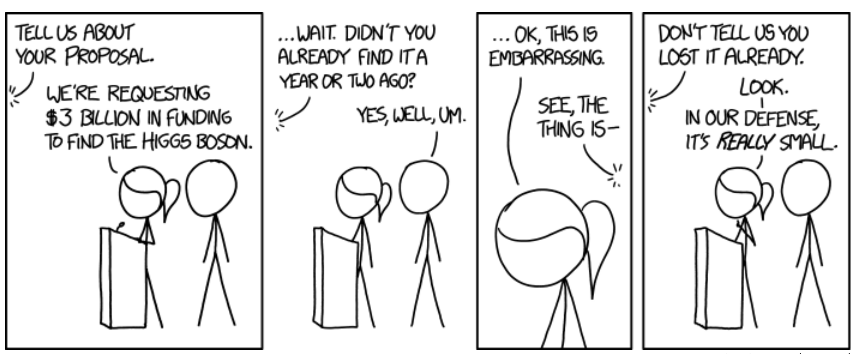


xkcd.com/1437/

Contents

1 Nuclear & Particle Physics

1.1 Radioactive Decay Decay processes are probabilistic in nature, and so one can safely suppose that the rate of decay is proportional to the number of states available to decay:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\gamma N \tag{1}$$

The solution to this is

$$N(t) = N_0 e^{-\gamma t} = N_0 e^{-t/\tau} = N_0 2^{-t/t_{1/2}} \tag{2}$$

γ is the decay rate, τ is the mean lifetime and $t_{1/2}$ is the half-life. They are related to each other by

$$\tau = \frac{1}{\gamma} = \frac{t_{1/2}}{\log 2} \tag{3}$$

After each half-life one expects half of a sample to have decayed.

1.2 Fission & Fusion Fission is the process through which nuclei split into smaller pieces. Unstable isotopes spontaneous undergo fission, and some may be excited through collisions to split. Uranium-235 bombarded by neutrons may undergo the fission reaction

$$n + {}^{235}_{92}\mathrm{U} \longrightarrow {}^{141}_{56}\mathrm{Ba} + {}^{92}_{36}\mathrm{Kr} + 3n \tag{4}$$

Because of the increase in the number of free neutrons, this leads to a cascade and the release of a large amount of energy.

Fusion is the opposite process, where nuclei combine to release energy. The binding energy per nucleon increases through the periodic table up until iron, at which the fusion process is not energetically favorable.

An important fusion process is the proton-proton chain reaction which is the first step in solar fusion. Protons fuse to form deuterium and then Helium-3. The steps are

$$^1\mathrm{H} + {}^1\mathrm{H} \longrightarrow {}^2\mathrm{He} + \gamma \tag{5}$$

$$^2\mathrm{He} \longrightarrow {}^2\mathrm{H} + e^+ + \nu_e \tag{6}$$

$$^2\mathrm{H} + {}^1\mathrm{H} \longrightarrow {}^3\mathrm{He} + \gamma \tag{7}$$

From here there are several possibilities:

- Branch I.

$$^3\mathrm{He} + {}^3\mathrm{He} \longrightarrow {}^4\mathrm{He} + 2 \cdot {}^1\mathrm{H} \tag{8}$$

- Branch II.

$$^3\mathrm{He} + {}^4\mathrm{He} \longrightarrow {}^7\mathrm{Be} + \gamma \tag{9}$$

$$^7\mathrm{Be} + e^- \longrightarrow {}^7\mathrm{Li} + \nu_e \tag{10}$$

$$^7\mathrm{Li} + {}^1\mathrm{H} \longrightarrow 2 \cdot {}^4\mathrm{He} \tag{11}$$

- Branch III.

$$^3\mathrm{He} + {}^4\mathrm{He} \longrightarrow {}^7\mathrm{Be} + \gamma \tag{12}$$

$$^7\mathrm{Be} + {}^1\mathrm{H} \longrightarrow {}^8\mathrm{B} + \gamma \tag{13}$$

$$^8\mathrm{B} \longrightarrow {}^8\mathrm{Be} + e^+ + \nu_e \tag{14}$$

$$^8\mathrm{Be} \longrightarrow 2 \cdot {}^4\mathrm{He} \tag{15}$$

	Symbol	Antiparticle	Charge (e)
up	u	$\bar{u}$	$+2/3$
down	d	$\bar{d}$	$-1/3$
charm	c	$\bar{c}$	$+2/3$
strange	s	$\bar{s}$	$-1/3$
top	t	$\bar{t}$	$+2/3$
bottom	b	$\bar{b}$	$-1/3$

Figure 1: Three generations of quarks.

1.3 Elementary Particles The broadest classification of particles is into bosons (integral spin) and fermions (half-integral spin). Elementary fermions are categorized further into quarks and leptons. There are lepton numbers associated with each generation, and these quantities are conserved. For example, both electrons and electron neutrinos have electronic number $L_e = 1$, while their antiparticles have $L_e = -1$; all other particles have $L_e = 0$. In addition, antiparticle have the opposite sign for electric charge. One might imagine that since there is a conservation law for each generation of leptons that there would be a conservation law for each generation of quarks, i.e. upness and strangeness are conserved. However, **this is not the case**, and processes such as

$$\Lambda \to p^+ + \pi^- \qquad \Lambda = uds \qquad p^+ = uud \qquad \pi^- = \overline{u}d \tag{16}$$

are observed. Here the "underlying process" here is the conversion of a strange quark into an up/anti-up quark pair and one down quark.

Each quark may come in one of three colors: red, green or blue. Color confinement is an explanation as to why isolated quarks are not observed: states must overall be "colorless". Composite particles known as hadrons form what was sarcastically called the "particle zoo" before the discovery of their quark building blocks. Hadrons consists of baryons, made from three quarks, and mesons, made from a quark and antiquark. An emphasized particle is the J/ψ meson, which has quark content $c\bar{c}$. It provided the first strong evidence that the current three-quark model (only u, d and s) was incomplete.

	Symbol	Antiparticle	Charge (e)
electron	e^-	e^+	-1
electron neutrino	ν_e	$\bar{\nu}_e$	0
muon	μ^-	μ^+	-1
muon neutrino	ν_μ	$\bar{\nu}_\mu$	0
tau	τ^-	τ^+	-1
tau neutrino	ν_τ	$\bar{\nu}_\tau$	0

Figure 2: Three generations of leptons.

1.4 Symmetries Noether's theorem states that symmetries of a Lagrangian correspond to conserved quantities. There are two broad categories: continuous and discrete symmetries. Continuous symmetries include translations and rotations.

Three common discrete symmetries are charge conjugation, parity and time reversal, often referred to as C, P and T. Charge conjugation involves reversing the signs of all internal quantum numbers but keeping mass, energy, momentum and spin untouched. Parity changes the handedness of the coordinate system. Time reversal sends $t \to -t$. The strong and electromagnetic forces are

	Symbol	Antiparticle	Charge (e)	Spin	Interaction
photon	γ	self	0	1	Electromagnetism
$W^{\pm}$ boson	$W^{\pm}$	$W^{\mp}$	± 1	1	Weak interaction
Z boson	Z	self	0	1	Weak interaction
gluon	g	self	0	1	Strong interaction
Higgs boson	H^0	self	0	0	Mass
graviton	G	self	0	2	Gravitation

Figure 3: Gauge bosons, along with the Higgs boson and theorized graviton.

invariant under any one of C, P or T. It was assumed that all interactions are unchanged under these operations, but it was found that the weak force does not obey charge conjugation: only left-handed neutrinos and right-handed antineutrinos interact through the weak force. While the weak force seems to obey the combination CP at first glance, this higher symmetry is violated by kaon decay. It is strongly believed that all processes, however, are invariant under CPT.

Symmetry	Conserved Quantity
Time translation	Energy
Spacial translation	Linear momentum
Spacial rotation	Angular momentum
Gauge transformation	A charge, e.g. electric charge

2 Condensed Matter

2.1 Crystal Structure Atoms in a crystalline structure form a regular pattern which may be characterized into several categories known as Bravais lattices. In three dimensions there are 14 Bravais lattices, often depicted by its unit cell, which come in several “types”:

- Primitive: lattice points are only on the cell corners.
- Body-Centered: lattice points are on the cell corners and there is an additional point at the center of the cell.
- Face-Centered: lattice points are on the cell corners and there is an additional point at the center of each of the faces.
- Base-Centered: lattice points are on the cell corners and there is an additional point at the center of two opposite faces.

The primitive cell for a Bravais lattice has a number density of one. The volume of the primitive cell is the volume of the unit cell divided by its number density. Lattice points on the faces, edges and vertices are “shared” with adjacent cells. So, for example, a cube with lattice points only at its vertices has number density one and so is primitive.

2.2 X-Ray Diffraction Light incident on a regular lattice will interact with the atoms in the lattice and produce diffraction patterns. In order to obtain a maximum in intensity the path length taken must differ by an integer multiple of the wavelength. With interatomic distances d and incident angle θ (measured from the plane of the crystal), the points of constructive interference occur where

$$2d\sin\theta = n\lambda \qquad n \in \{1, 2, 3, \ldots\} \tag{17}$$

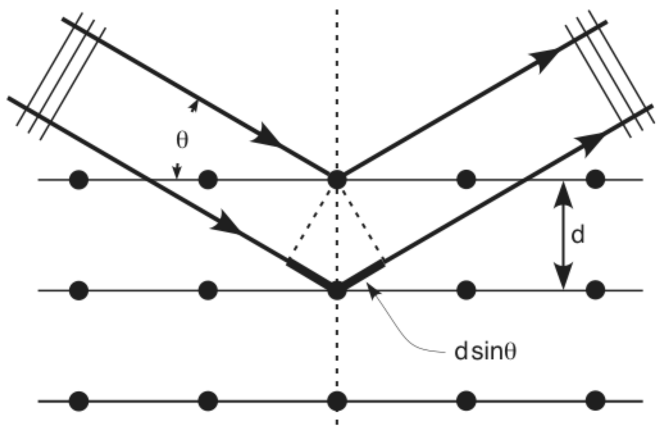


Figure 4: Bragg diffraction geometry

2.3 Semiconductors In materials with many atoms the electronic energy levels are warped from their clean, isolated-atom form to create energy bands. In a semiconductor the filled states and conducting states are separated by an energy gap that prohibits conduction unless an electron is promoted after absorbing the energy of a photon. The resistance of a semiconductor decreases as its temperature increases, as the increased thermal energy is more readily able to promote electrons to the conduction band. This is in contrast to metals, for which the resistance increases with increasing temperature.

The process of doping consists of introducing impurities to the semiconductor to increase the number of charge carriers. Adding electron donors results in a net negative charge and is called "n-type", and adding electron acceptors results in a net positive charge and is called "p-type".

2.4 Superconductors At low enough temperatures the electrical resistance of many materials drops exactly to zero; this is the phenomenon known as superconductivity. The point at which the resistance jumps quickly to zero is known as the critical temperature. With no resistance current will flow indefinitely even with no power source. Magnetic fields are expelled from the superconductor; this is the Meissner effect.

BCS theory explains the macroscopic phenomenon of superconductivity by supposing that electrons form what are known as Cooper pairs. At low enough temperatures two electrons may condense into a bosonic bound state, and these bosons are no longer restricted by the Pauli exclusion principle.

3 Mathematics

3.1 Curvilinear Coordinate Systems The notation $S_x = \sin x$ and $C_x = \cos x$ is used in the following table. Polar coordinates are cylindrical with the restriction $z = 0$, where the relevant measure is the area, given by $\mathrm{d}A = r\,\mathrm{d}r\mathrm{d}\theta$.

	Cylindrical	Spherical
Distance Element	$\mathrm{d}\boldsymbol{s} = \mathrm{d}r\,\hat{\boldsymbol{r}} + r\,\mathrm{d}\theta\,\hat{\boldsymbol{\theta}} + \mathrm{d}z\,\hat{\boldsymbol{z}}$	$\mathrm{d}\boldsymbol{s} = \mathrm{d}r\,\hat{\boldsymbol{r}} + r\,\mathrm{d}\theta\,\hat{\boldsymbol{\theta}} + r\,\sin\theta\,\mathrm{d}\phi\hat{\boldsymbol{\phi}}$
Volume Element	$\mathrm{d}V = r\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$	$\mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$
Unit Vectors	$\begin{bmatrix}\hat{\boldsymbol{r}}\\ \hat{\boldsymbol{\theta}}\\ \hat{\boldsymbol{z}}\end{bmatrix} = \begin{bmatrix}C_\phi & S_\phi & 0\\ -S_\phi & C_\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{\imath}}\\ \hat{\boldsymbol{\jmath}}\\ \hat{\boldsymbol{k}}\end{bmatrix}$	$\begin{bmatrix}\hat{\boldsymbol{r}}\\ \hat{\boldsymbol{\theta}}\\ \hat{\boldsymbol{\phi}}\end{bmatrix} = \begin{bmatrix}S_\theta C_\phi & S_\theta S_\phi & C_\theta\\ C_\theta C_\phi & C_\theta S_\phi & -C_\theta\\ -S_\phi & C_\phi & 0\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{\imath}}\\ \hat{\boldsymbol{\jmath}}\\ \hat{\boldsymbol{k}}\end{bmatrix}$
Kinetic Energy	$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right)$	$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right)$

3.2 Vector Differential Operators

Gradient $\boldsymbol{\nabla}$	$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}}$ $\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$ $\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}$
Divergence $\boldsymbol{\nabla}\cdot$	$\boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$ $\boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{1}{r}\frac{\partial}{\partial r}\big(rF_r\big) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$ $\boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2F_r\big) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\big(\sin\theta\,F_\theta\big) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$
Curl $\boldsymbol{\nabla}\times$	$\boldsymbol{\nabla}\times\boldsymbol{F} = \begin{vmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_x & F_y & F_z\end{vmatrix}$ $\boldsymbol{\nabla}\times\boldsymbol{F} = \frac{1}{r}\begin{vmatrix}\hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & \hat{\boldsymbol{z}}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z}\\ F_r & rF_\theta & F_z\end{vmatrix}$ $\boldsymbol{\nabla}\times\boldsymbol{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi}\\ F_r & rF_\theta & r\sin\theta F_\phi\end{vmatrix}$
Laplacian $\nabla^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ $\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$ $\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2 f}{\partial \phi^2}$

These vector operators satisfy several identities:

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla} f) = 0 \tag{18}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{F}) = 0 \tag{19}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{F}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{F}) - \nabla^2\boldsymbol{F} \tag{20}$$

The Helmholtz decomposition theorem states that a sufficiently smooth vector field may be written as the sum of a gradient and a curl:

$$\boldsymbol{F} = -\boldsymbol{\nabla}\Phi + \boldsymbol{\nabla}\times\boldsymbol{M} \tag{21}$$

In particular, if $\boldsymbol{F}$ is divergence-free, then it may be written

$$\boldsymbol{F} = \boldsymbol{\nabla}\times\boldsymbol{M} \tag{22}$$

and if $\boldsymbol{F}$ is curl-free, then it may be written

$$\boldsymbol{F} = -\boldsymbol{\nabla}\Phi \tag{23}$$

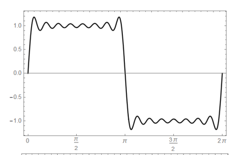
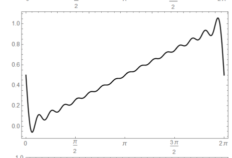
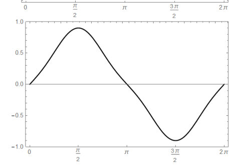
3.3 Fourier Series Periodic functions may be decomposed into a sum of sines and cosines. Assuming a period of 2π for a function f, we have the decomposition

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos nx + b_n \sin nx\big) \tag{24}$$

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos mx \, \mathrm{d}x \qquad m \in \{0, 1, 2, \ldots\} \tag{25}$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx \, \mathrm{d}x \qquad k \in \{1, 2, 3, \ldots\} \tag{26}$$

Some common examples of Fourier series are given below:

Square Wave	$f(x) = \frac{4}{\pi} \sum\limits_{n \text{ odd}} \frac{1}{n} \sin nx$	
Saw-Tooth	$f(x) = \frac{1}{2} - \frac{1}{\pi} \sum\limits_{n=1}^{\infty} \frac{1}{n} \sin nx$	
Triangle Wave	$f(x) = \frac{8}{\pi^2} \sum\limits_{n \text{ odd}} \frac{(-1)^{(n-1)/2}}{n^2} \sin nx$	

Things to look for:

- Parity: Is the expression even or odd?
- Special values: Where is the sum obviously zero, positive or negative?
- Translational symmetry: If the function is shifted does it flip parity?
- Convergence rates: If the $(k-1)^{\text{th}}$ derivative is continuous and the k^{th} derivative is discontinuous then the coefficients go as $a_n \sim \frac{1}{n^{k+1}}$.

3.4 Matrix Algebra Matrices provide a convenient way to solve coupled systems and differential equations. In addition, one formulation of Quantum mechanics is matrix mechanics, in which particle states are represented by matrices. It is equivalent to the Schrödinger wave formulation, but provides a different mechanism for solving problems.

The determinant of a matrix a useful tool for linear algebra. Up to a sign it represents the n-volume of the parallelepiped determined by the columns or rows (viewed as vectors in $\mathbb{R}^n$). There are several ways to compute a determinant for a square matrix, including expansion by minors and summing over permutations. In the case of 2×2 matrices it is particularly straightforward:

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{27}$$

Determinants are multiplicative, so that $\det AB = \det A \det B$. In particular, $\det A^{-1} = \frac{1}{\det A}$. A matrix has an inverse iff its determinant is nonzero. In the 2×2 case the inverse of a matrix is

$$A^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \tag{28}$$

Another useful function is the trace, which is the sum of the diagonal elements. It is not multiplicative, but is additive and cyclic, so that

$$\operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B) \qquad \operatorname{tr}(ABC) = \operatorname{tr}(BCA) = \operatorname{tr}(CAB) \tag{29}$$

These two functions show their use when considering the eigenvalue expression $AX = \lambda X$. Rearranging gives $(\lambda\mathbb{I} - A)X = 0$. If $\lambda\mathbb{I} - A$ has an inverse, then we may multiply both sides by $(\lambda\mathbb{I} - A)^{-1}$ to arrive at $X = 0$. Since this is boring and obviously a solution, we consider the case where $\lambda\mathbb{I} - A$ is not invertable. This occurs exactly when $\det(\lambda\mathbb{I} - A) = 0$. This gives a polynomial in λ with coefficients determined by the elements in A: the characteristic polynomial for A, whose roots are the eigenvalues of A.

$$\det(\lambda\mathbb{I} - A) = \lambda^n - \operatorname{tr}(A)\lambda^{n-1} + \cdots + (-1)^n \det(A) \tag{30}$$

On the other hand, if the polynomial has roots $\{\lambda_i\}$, counting multiplicities, we have the factorization

$$\begin{aligned} \det(\lambda\mathbb{I} - A) &= (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) && (31) \\ &= \lambda^n - (\lambda_1 + \cdots + \lambda_n)\lambda^{n-1} + \cdots + (-1)^n(\lambda_1 \cdots \lambda_n) && (32) \end{aligned}$$

Comparing these two approaches shows that we have

$$\det A = \prod_{i=1}^{n} \lambda_i \qquad \operatorname{tr} A = \sum_{i=1}^{n} \lambda_i \tag{33}$$

Again, in the 2×2 case this proves to be quite useful. Denoting the two eigenvalues of A by $\lambda_\pm$, we have

$$\lambda_+\lambda_- = \det A \qquad \lambda_+ + \lambda_- = \operatorname{tr} A \qquad \Longrightarrow \qquad \lambda_\pm = \tfrac{\operatorname{tr} A}{2} \pm \sqrt{\left(\tfrac{\operatorname{tr} A}{2}\right)^2 - \det A} \tag{34}$$

A Summary

Radioactive Decay

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\gamma N \qquad \text{(Exponential Decay)}$$

$$N(t) = N_0 e^{-\gamma t} = N_0 e^{-t/\tau} = N_0 2^{-t/t_{1/2}}$$

$$\tau = \frac{1}{\gamma} = \frac{t_{1/2}}{\log 2}$$

X-Ray Diffraction

$$2d\sin\theta = n\lambda \qquad n \in \{1, 2, 3, \ldots\} \qquad \text{(Bragg Diffraction)}$$

Vector Differential Operators

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}} \qquad \text{(Gradient)}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \qquad \text{(Divergence)}$$

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\boldsymbol{\imath}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\boldsymbol{k}} \qquad \text{(Curl)}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \qquad \text{(Laplacian)}$$

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \qquad \text{(Fourier Series)}$$

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos mx\,\mathrm{d}x \qquad m \in \{0, 1, 2, \ldots\}$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,\mathrm{d}x \qquad k \in \{1, 2, 3, \ldots\}$$

Matrix Algebra

$$\operatorname{tr} A = \sum_i a_{ii} = \sum_i \lambda_i \qquad \text{(Trace)}$$

$$\det A = \prod_i \lambda_i \qquad \text{(Determinant)}$$

$$\det A_{2\times 2} = a_{11}a_{22} - a_{12}a_{21} \qquad (2 \times 2 \text{ Determinant})$$

$$A_{2\times 2}^{-1} = \frac{1}{\det A}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \qquad (2 \times 2 \text{ Inverse})$$

$$\lambda_\pm = \frac{\operatorname{tr} A}{2} \pm \sqrt{\left(\frac{\operatorname{tr} A}{2}\right)^2 - \det A} \qquad (2 \times 2 \text{ Eigenvalues})$$