Physics GRE:

Special & General Relativity

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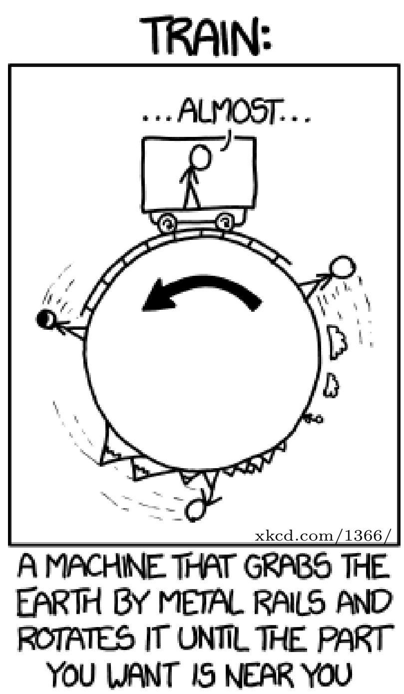

Contents

1 Special Relativity

1.1 Four-Vectors & Lorentz Transformations The special theory of relativity stems from two ideas: the laws of physics are the same in any inertial frame and the speed of light in a vacuum is the same for all observers. These two assumptions together have drastic consequences. Space and time are now roughly on the same level, and so it is quite common to introduce a new notion for coordinates:

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) \qquad x_\mu = \eta_{\mu\nu} x^\nu = (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z) \tag{1}$$

The Minkowski metric, $\eta_{\mu\nu}$, is here given by $\mathrm{diag}\,(1, -1, -1, -1)$; some use the negative of this.

Two observers will often not agree on the order of events, lengths or times, but all observers will agree on several things: so-called Lorentz invariants. One is, of course, the speed of light. Then there is the spacetime interval,

$$\mathrm{d}s^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = c^2\mathrm{d}t^2 - \left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) = c^2\mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2 \tag{2}$$

Intervals between two events are categorized by their sign:

- Time-like ($\Delta s^2 > 0$): The time component of Δs^2 dominates. There exists an inertial frame where the two events occur at the same point in space.
- Light-like ($\Delta s^2 = 0$): A photon emitted at event one would arrive at event two.
- Space-like ($\Delta s^2 < 0$): The space component of Δs^2 dominates. There exists an inertial frame where the two events are simultaneous.

Lorentz transformations are those transformations which leave the spacetime interval invariant by transforming one inertial frame into another. This is a change of coordinates, and so is given by

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{3}$$

where a^μ is a constant, and to maintain the spacetime interval we require that Λ satisfy

$$\Lambda^\mu{}_\sigma \eta^{\sigma\rho} \Lambda^\nu{}_\rho = \eta^{\mu\nu} \tag{4}$$

These include the usual spacial rotations and so-called boosts, which take a frame S into a frame S' moving with relative speed v. We may as well assume that the boost occurs along the x-axis (rotate if not) and that the origins coincide at $t = 0$. Then spacetime coordinates in S' are related to those in S by

$$\begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{v}{c} \tag{5}$$

In perhaps a more familiar form, these are

$$t' = \gamma(t - vx/c^2) \qquad x' = \gamma(x - vt) \qquad y' = y \qquad z' = z \tag{6}$$

Composition of boosts along the same axis correspond to velocity addition. If an observer A sees an observer B moving with velocity v along the x-axis, and in a Lorentz frame where B is stationary there is an object C moving with velocity u along the x-axis, what velocity does A measure C to be going? That is, if B is moving with velocity v relative to A and C is moving with velocity u relative to B, with what velocity is C moving relative to A? The result is

$$v' = \frac{v+u}{1+\frac{vu}{c^2}} \tag{7}$$

Note that if both v and u are less than c, then so is v'. One may also express the velocity in terms of a new parameter, θ, called the rapidity:

$$\beta = \tanh\theta \tag{8}$$

This simplifies calculations involving velocity addition, since we have

$$\beta' = \frac{\tanh\theta_1 + \tanh\theta_2}{1+\tanh\theta_1\tanh\theta_2} = \tanh\left(\theta_1+\theta_2\right) \tag{9}$$

That is, rapidities add. Using this new quantity, Lorentz transformations take a simple form:

$$\begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{10}$$

Expressions for momentum and energy must be altered:

$$\boldsymbol{p} = \gamma m \boldsymbol{v} \qquad E = \gamma m c^2 \tag{11}$$

For small $\boldsymbol{v}$ these are

$$\boldsymbol{p} = m\boldsymbol{v} + \mathcal{O}(v^3) \qquad E = mc^2 + \frac{1}{2}mv^2 + \mathcal{O}(v^4) \tag{12}$$

agreeing with the nonrelativistic notions of momentum and energy. The constant term for E is known as the rest energy. Eliminating $\boldsymbol{v}$ in their definitions shows that these relativistic quantities satisfy

$$E^2 = (mc^2)^2 + |\boldsymbol{p}|^2c^2 \tag{13}$$

This now makes sense for massless particles as well.

1.2 Time Dilation & Length Contraction From the spacetime interval we may also construct the proper time interval:

$$\mathrm{d}\tau^2 = \frac{\mathrm{d}s^2}{c^2} = \mathrm{d}t^2 - \frac{1}{c^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) \tag{14}$$

Two events separated by a time-like interval satisfy

$$\Delta\tau^2 = \Delta t^2 - \frac{\Delta\boldsymbol{x}^2}{c^2} > 0 \qquad \Longrightarrow \qquad \Delta t = \sqrt{\Delta\tau^2 + \frac{\Delta\boldsymbol{x}^2}{c^2}} \geq \Delta\tau \tag{15}$$

That is, the time interval between two events is minimized in a Lorentz frame where the two events occur at the same point in space. This is time dilation. Any observer moving relative to a ticking clock will measure it to be ticking more slowly than a timepiece at their side. For an observer moving with relative velocity v the measured time is

$$\Delta t = \gamma \Delta\tau = \frac{\Delta\tau}{\sqrt{1-\beta^2}} \tag{16}$$

In a similar way the relative motion dilates time, relative motion distorts lengths. A rod of length L_0 as measured at rest will be measured to have a length

$$L = \frac{L_0}{\gamma} = L_0\sqrt{1-\beta^2} \tag{17}$$

if boosted in the direction of its extent. Distances are contracted along the direction of relative motion, but orthogonal distances are unaffected.

A prime example of these effects is in atmospheric muon decay. Cosmic rays produce muons at the top of the atmosphere, a height of about $h_0 = 10\,\text{km}$. At rest muons have a half-life of $t_0 = 1.56\cdot 10^{-6}\,\text{s}$. These muons are highly relativistic, travelling with a speed about $v = 0.98c$. Naively applying non-relativistic equations predicts that only $0.3\cdot 10^{-6} = \frac{3}{10^7}$ of the produced muons will reach the Earth's surface: far fewer than is observed.

From the muon's perspective the Earth is rushing up to meet it, and so the distance to the Earth is length contracted by a factor of $\gamma \approx 5$. The number of half-lives passing is then

$$\frac{t}{t_0} = \frac{(h_0/\gamma)}{vt_0} \approx \frac{0.2(10^4\,\text{m})}{(0.98\cdot 3\cdot 10^8\,\text{m/s})(1.56\cdot 10^{-6}\,\text{s})} \approx 4.36 \tag{18}$$

This means that the proportion of muons reaching the surface of the Earth is approximately $2^{-4.36} \approx \frac{1}{20.5}$. From the perspective of an observer on the Earth, the muons are moving very quickly and will suffer from time dilation. The muons' internal clocks will appear to run slower by a factor of $\gamma \approx 5$. The number of half-lives passign is then

$$\frac{t}{t_0} = \frac{h_0}{v(\gamma t_0)} \approx 4.36 \tag{19}$$

The two observers agree on the number of half-lives that pass, but disagree on the reasoning that leads to the inclusiong of a factor of five.

1.3 Simultaneity Two events that occur at the same time in one inertial frame will necessarily be measured to occur at different times in any other inertial frame. This might seem to lead to paradoxical situations, but all observers will agree on "absolute" characteristics of events: anything they could meet up later and agree upon. Undoubtedly they will disagree on the explanation for such occurrences, but the ultimate outcomes must be the same. A prime example of this is of a ladder passing through a barn of exactly the same length with two open doors. An observer stationary off to the side will see a length contracted ladder fit nicely within the barn. An observer on the ladder will see a length contracted barn so that the ladder is never completely inside the barn. Now the barn-observer quickly closes and then opens the doors when he sees the back-end of the ladder disappear within the barn. What does the ladder-observer see? The two observers must agree that the ladder is not hit by the doors, and so we must conclude that the ladder-observer does not see the two doors close and open at the same time: simultaneity is lost!

1.4 Energy & Momentum

Time and space together form a 4-vector. So too do energy and momentum:

$$p^\mu = (p^0, p^1, p^2, p^3) = \left(\tfrac{E}{c}, p_x, p_y, p_z\right) \qquad p_\mu = \eta_{\mu\nu} p^\nu = (p_0, p_1, p_2, p_3) = \left(\tfrac{E}{c}, -p_x, -p_y, -p_z\right) \tag{20}$$

Invariance of Lorentz scalars gives that all observers will agree on the value of $p_\mu p^\mu$:

$$p_\mu p^\mu = \eta_{\mu\nu} p^\mu p^\nu = \frac{E^2}{c^2} - \left(p_x^2 + p_y^2 + p_z^2\right) = \frac{E^2}{c^2} - |\boldsymbol{p}|^2 \tag{21}$$

Using the above relation between E and $\boldsymbol{p}$ we see that the invariant quantity is proportional to the rest mass:

$$p_\mu p^\mu = m^2 c^2 \tag{22}$$

1.5 Electromagnetism

The electric potential and vector potential may be combined to form the electromagnetic four-potential:

$$A^\mu = \left(\tfrac{V}{c}, A_x, A_y, A_z\right) = \left(\tfrac{V}{c}, \boldsymbol{A}\right) \tag{23}$$

Maxwell's equations may be expressed in terms of the electromagnetic field tensor, given by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix} \tag{24}$$

It transforms under Lorentz transformations as

$$F'^{\mu\nu} = \Lambda^\mu{}_\sigma \Lambda_\rho{}^\nu F^{\sigma\rho} \tag{25}$$

As seen by the components of $F^{\mu\nu}$ above, these transform electric and magnetic fields into one-another. An electric field is never completely transformed into a magnetic field for any boost, as it would naively require boosting the source up to the speed of light: not a Lorentz transformation. The components of the fields parallel and orthogonal to the boost transform as

$$\boldsymbol{E}'_{||} = \boldsymbol{E}_{||} \qquad \boldsymbol{E}'_\perp = \gamma(\boldsymbol{E}_\perp + \boldsymbol{v} \times \boldsymbol{B}_\perp) \tag{26}$$

$$\boldsymbol{B}'_{||} = \boldsymbol{B}_{||} \qquad \boldsymbol{B}'_\perp = \gamma(\boldsymbol{B}_\perp - \boldsymbol{v} \times \boldsymbol{E}_\perp / c^2) \tag{27}$$

For example, a boost along the x-axis gives

$$E'_x = E_x \qquad E'_y = \gamma(E_y - vB_z) \qquad E'_z = \gamma(E_z + vB_y) \tag{28}$$

$$B'_x = B_x \qquad B'_y = \gamma(B_y + vE_z/c^2) \qquad B'_z = \gamma(B_z - vE_y/c^2) \tag{29}$$

1.6 Doppler Effect

Relativistic doppler shift is

$$\lambda = \lambda_0 \sqrt{\frac{1 \pm \beta}{1 \mp \beta}} \qquad f = f_0 \sqrt{\frac{1 \mp \beta}{1 \pm \beta}} \tag{30}$$

The sign is determined by whether the source is moving towards or away from the detector: if towards then the detected frequency will be higher and the wavelength lower.

2 General Relativity

Solving the Einstein field equations for a stationary, spherically symmetric, electrically neutral mass distribution (boring!) gives the Schwarzschild solution. There arises a distinguished radius, known as the Schwarzschild radius, which, for an object of mass M, is

$$r_{\mathrm{s}} = \frac{2GM}{c^2} \tag{31}$$

If all of the mass is concentrated within this radius then the object is a black hole.

Object	Mass	r_{s}
Sun	$2.0 \cdot 10^{30}\,\mathrm{kg}$	$3.0\,\mathrm{km}$
Earth	$6.0 \cdot 10^{24}\,\mathrm{kg}$	$8.9\,\mathrm{mm}$
Mt. Everest	$\sim 6.0 \cdot 10^{15}\,\mathrm{kg}$	$8.9\,\mathrm{pm}$

A Summary

Four-Vectors & Lorentz Transformations

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) \qquad \text{(Four-Position)}$$

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \operatorname{diag}(1, -1, -1, -1) \qquad \text{(Minkowski Metric)}$$

$$M_\mu = \eta_{\mu\nu} M^\nu \qquad M^\mu = \eta^{\mu\nu} M_\nu \qquad \text{(Raising \& Lowering of Indices)}$$

$$\mathrm{d}s^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu = c^2\mathrm{d}t^2 - \left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) = c^2\mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2 \qquad \text{(Spacetime Interval)}$$

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \frac{1}{\sqrt{1 - \beta^2}} \qquad \text{(Lorentz Factor)}$$

$$\begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \qquad \text{(Boost Along } x\text{-Axis)}$$

$$v' = \frac{v + u}{1 + \frac{vu}{c^2}} \qquad \text{(Colinear Velocity Addition)}$$

$$\boldsymbol{p} = \gamma m \boldsymbol{v} \qquad \text{(Momentum)}$$

$$E = \gamma m c^2 \qquad \text{(Energy)}$$

$$E^2 = (mc^2)^2 + |\boldsymbol{p}|^2 c^2$$

$$p^\mu = (p^0, p^1, p^2, p^3) = (\tfrac{E}{c}, p_x, p_y, p_z) \qquad \text{(Four-Momentum)}$$

$$p_\mu p^\mu = m^2 c^2$$

Time Dilation & Length Contraction

$$\Delta t = \gamma \Delta \tau \qquad \text{(Time Dilation)}$$

$$L = \frac{L_0}{\gamma} \qquad \text{(Length Contraction)}$$

Electromagnetism

$$A^\mu = (A^0, A^1, A^2, A^3) = (\tfrac{V}{c}, A_x, A_y, A_z) \qquad \text{(Four-Potential)}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \qquad \text{(Electromagnetic Field Tensor)}$$

$$\begin{aligned} E'_x &= E_x & B'_x &= B_x \qquad \text{(Boost Along } x\text{-Axis)} \\ E'_y &= \gamma(E_y - vB_z) & B'_y &= \gamma(B_y + vE_z/c^2) \\ E'_z &= \gamma(E_z + vB_y) & B'_z &= \gamma(B_z - vE_y/c^2) \end{aligned}$$

Doppler Effect

$$\lambda = \frac{c}{f} = \lambda_0 \sqrt{\frac{1 \pm \beta}{1 \mp \beta}} = \frac{c}{f_0} \sqrt{\frac{1 \pm \beta}{1 \mp \beta}} \qquad \text{(Doppler Effect)}$$

General Relativity

$$r_\mathrm{s} = \frac{2GM}{c^2} \qquad \text{(Schwarzchild Radius)}$$